Bayesian filtering for quantum-enhanced atomic sensors

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Bayesian filtering for quantum-enhanced atomic sensors

Morgan Mitchell –
Quantum Information Theory with Cold Atoms and Non-classical Light

PHYSICAL REVIEW LETTERS **120,** 040503 (2018)

Signal Tracking Beyond the Time Resolution of an Atomic Sensor by Kalman Filtering

Ricardo Jiménez-Martínez,[1,*] Jan Kołodyński,[1] Charikleia Troullinou,[1] Vito Giovanni Lucivero,[1]
Jia Kong,[1] and Morgan W. Mitchell[1,2]

https://arxiv.org/abs/1707.08131

Atomic ensembles as sensors

<u>Key</u>: phenomenon of **optical pumping**

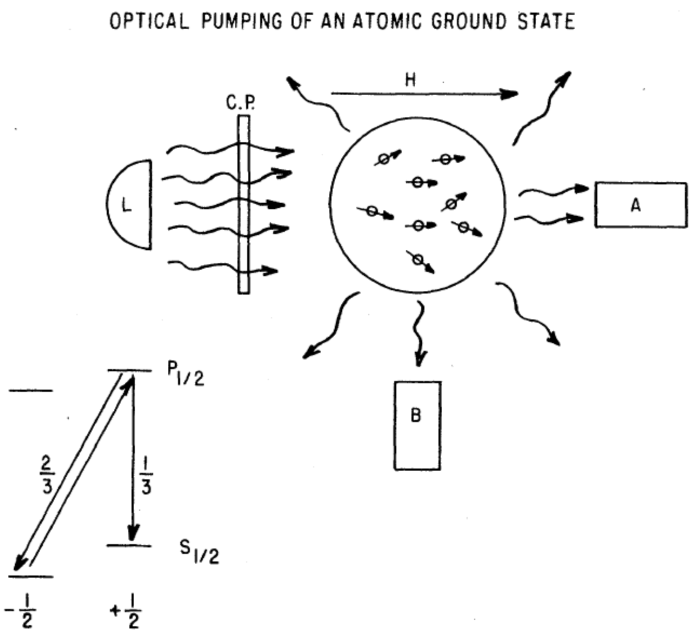


FIG. 1. A simple optical pumping experiment. Atoms are polarized by the scattering of circularly polarized resonant light. Either the transmitted light at A or the fluorescently scattered light at B can be used to monitor the atomic polarization.

REVIEWS OF MODERN PHYSICS VOLUME 44, NUMBER 2

Optical Pumping*

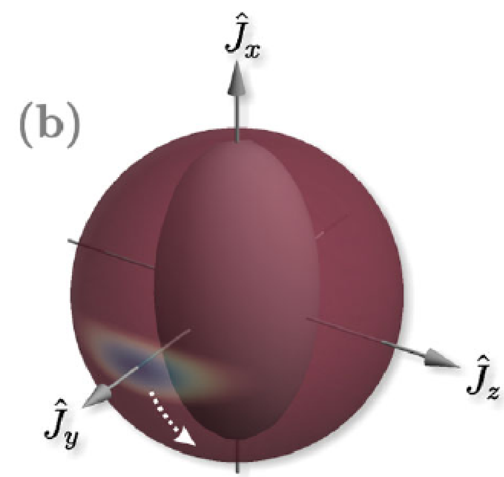
WILLIAM HAPPER
Columbia Radiation Laboratory, Department of Physics, Columbia University, New York, New York 10027

single-atom spin (nuclear+electronic parts):

$$\hat{\mathbf{j}} = \hat{\mathbf{I}} + \hat{\mathbf{S}}$$

collective **atomic-ensemble spin**:

$$\hat{\mathbf{J}} = \sum_n \hat{\mathbf{j}}^{(n)}$$



[Happer, Jau, Walker "*Optically Pumped Atoms*" (Wiley 2009)]

Atomic ensembles as sensors

<u>Key:</u> phenomenon of **optical pumping**

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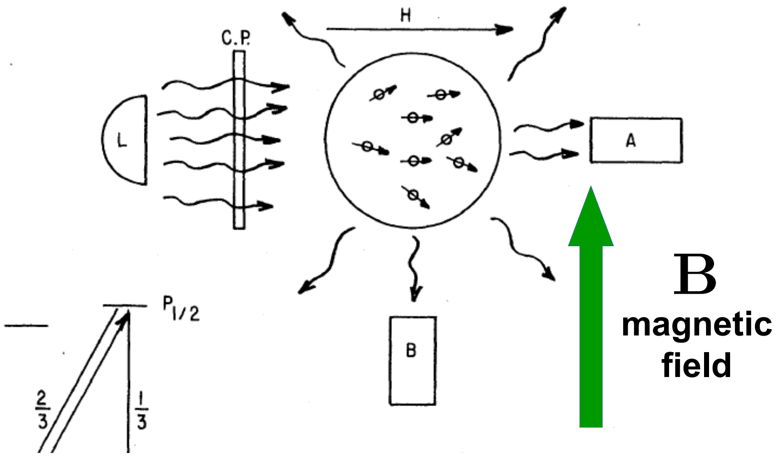
$$\hat{\mathbf{j}} = \hat{\mathbf{I}} + \hat{\mathbf{S}}$$

collective **atomic-ensemble spin**:

$$\hat{\mathbf{J}} = \sum_n \hat{\mathbf{j}}^{(n)}$$

OPTICAL PUMPING OF AN ATOMIC GROUND STATE



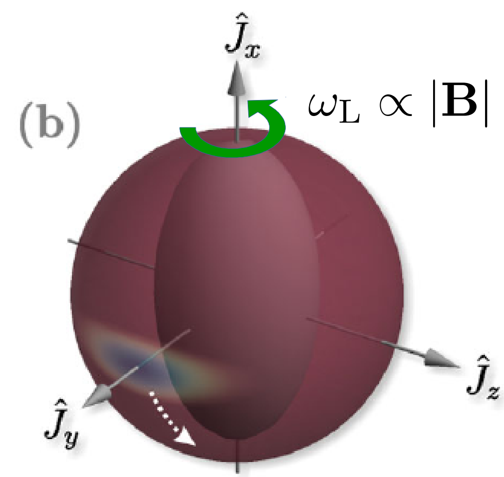
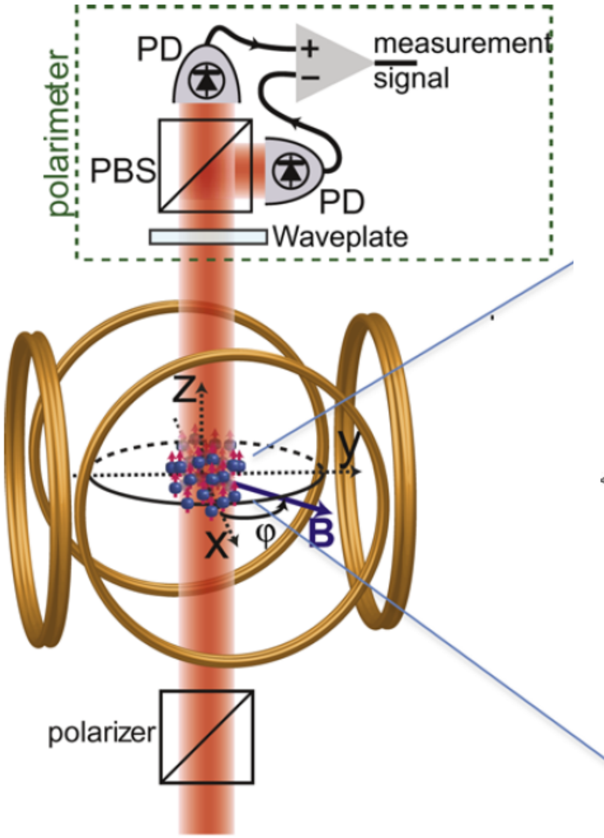


TABLE I
PROPERTIES OF SEVERAL TYPES OF ATOMIC SENSORS

Sensor type	T_2 [s]	$\delta\varphi$ [rad] @ T_2	H_{int}	$\omega_0/2\pi$	*Stability or Sensitivity @ 1s*
Ion optical clock	1	~1	$\alpha q_n q_e$	10^{15} Hz	10^{-15}
Fountain atomic clock	1	10^{-4}	$\beta \vec{\mu}_n \cdot \vec{\mu}_e$	10^{10} Hz	10^{-14}
Beam atomic clock	0.01	10^{-3}	$\beta \vec{\mu}_n \cdot \vec{\mu}_e$	10^{10} Hz	10^{-12}
Vapor cell magnetometer	0.01	10^{-4}	$\vec{\mu} \cdot \vec{B}$	10^{10} Hz/T	10^{-13} T
Vapor cell NMR gyroscope	100	10^{-8}	$\vec{L} \cdot \vec{\Omega}$	7×10^{-7} Hz/(°/h)	10^{-3} °/h
Laser cooled atom interferometer accel.	1	0.1	$\hbar k T a$	10^{7} Hz/g	10^{-8} g
Atomic electric field sensor	10^{-7}	~0.1	$\vec{d} \cdot \vec{E}$	10^{8} Hz/(V/cm)	~10 µV/cm

[Kitching et al, "*Atomic Sensors – A Review*" (IEEE Sensors 2011)]

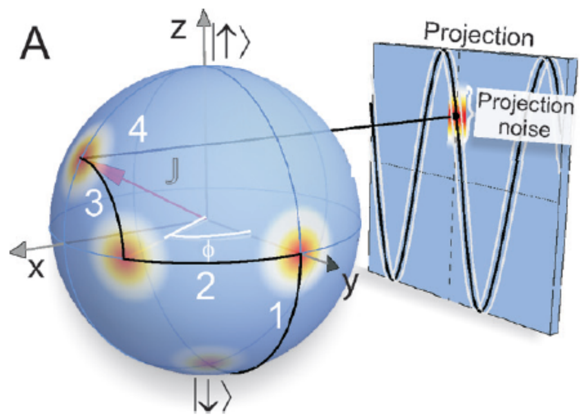
Conditional squeezing by QND (Faraday) measurements

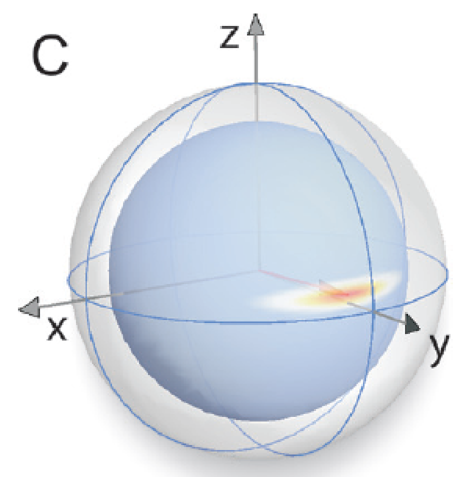


Polarisation of the probe light rotated by a **Faraday angle:**

$$\hat{\Theta}_{\mathrm{FR}} = g\,\hat{J}_z + \ldots$$

("weak" QND measurement, off/on-resonance)





(single-shot)
conditional squeezed (in J_z) **state**
after the QND measurement

[Deutsch I. and Jessen P., *Optics Communications* 283 (2010) 681–694]
[Hammerer et al, "Quantum interface between light and atomic ensembles" RMP 82 (2010)]

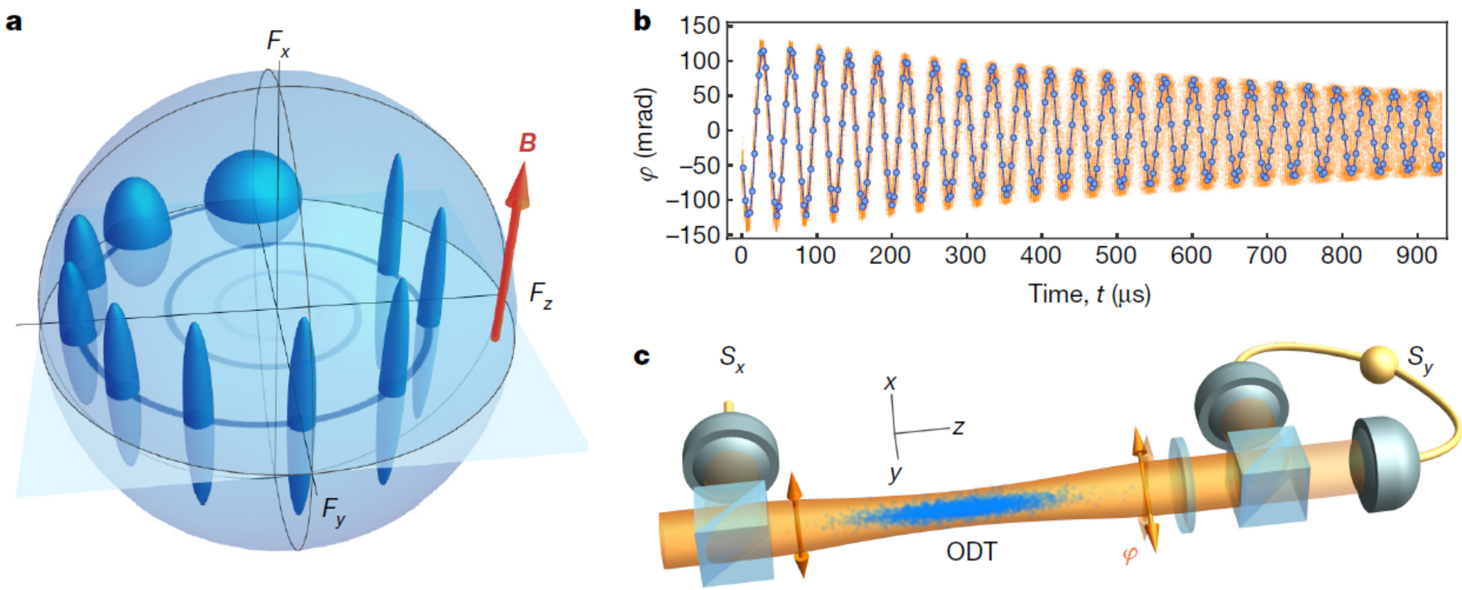
Conditional squeezing by QND (Faraday) measurements

LETTER

doi:10.1038/nature21434

Simultaneous tracking of spin angle and amplitude beyond classical limits

Giorgio Colangelo[1], Ferran Martin Ciurana[1], Lorena C. Bianchet[1], Robert J. Sewell[1] & Morgan W. Mitchell[1,2]



"***evading***" Heisenberg uncertainty relation:

$$\delta F_y \delta F_z \geq \frac{1}{2}|\langle [F_y, F_z] \rangle| = \frac{1}{2}|\langle F_x \rangle|$$

It is a *continuous measurement*, but can we use quantum continuous (stochastic) measurement formalism?

(with back-action!)

Continuous conditional spin-squeezing

VOLUME 91, NUMBER 25 PHYSICAL REVIEW LETTERS week ending 19 DECEMBER 2003

Quantum Kalman Filtering and the Heisenberg Limit in Atomic Magnetometry

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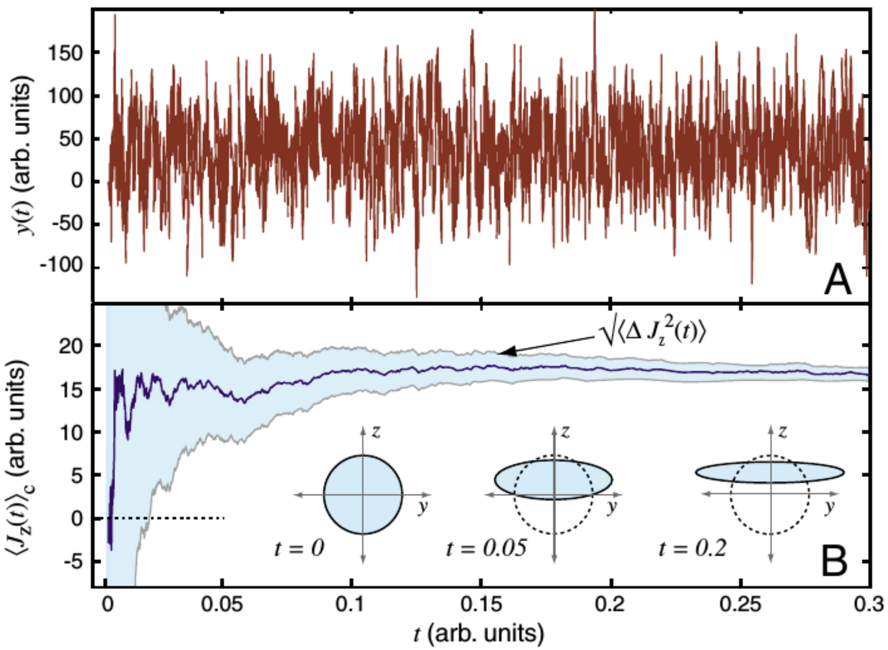
Norman Bridge Laboratory of Physics, California Institute of Technology, Pasadena, California, 91125, USA
(Received 27 June 2003; published 19 December 2003)



Quantum continuous measurement framework (with homodyne detection):

Measurement dynamics:

$$y(t)dt = 2\eta\sqrt{M}\langle \hat{J}_z(t)\rangle_c dt + \sqrt{\eta}\, dW(t),$$

Ensemble dynamics:

$$d\hat{\rho}_c(t) = -i\gamma B[\hat{J}_y, \hat{\rho}_c]dt + \boxed{M\mathcal{D}[\hat{J}_z]\hat{\rho}_c dt} + \sqrt{M\eta}\,\mathcal{H}[\hat{J}_z]\hat{\rho}_c dW(t),$$

BUT SUCH A MODEL IS NOT ACCURATE ENOUGH !!

Current best descriptions of atomic-ensemble dynamics (e.g., **SERF magnetometers**) rely on **single-atom models**:

$$\rho = \frac{1}{N}\sum_n \rho^{(n)} \quad \Rightarrow \quad \frac{d\rho}{dt} = a_{hf}\frac{[\mathbf{I}\cdot\mathbf{S},\rho]}{i\hbar} + \mu_B g_S \frac{[\mathbf{B}\cdot\mathbf{S},\rho]}{i\hbar} + \frac{\varphi(1+4\langle\mathbf{S}\rangle\cdot\mathbf{S})-\rho}{T_{SE}} + \frac{\varphi-\rho}{T_{SD}} + R[\varphi(1+2\mathbf{s}\cdot\mathbf{S})-\rho] + D\nabla^2\rho.$$

[Appelt et al, PRA 58(2), 1412 (1998); F. Grossetete, J. Phys. (Paris) 25, 383 (1964); 29, 456 (1968)]

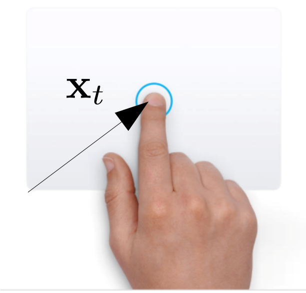
What to do then?... Go step by step!

STEP 1 [the experiment presented today...]

- Describe the **atomic spin-noise** phenomenologically (via *noise spectroscopy*) in a stochastic manner.
- Work in the regime in which "back-action" can be avoided, so that **photon shot-noise** is also stochastic and **independent** from the atomic noise.
- Sense stochastic **input signals** *(waveform estimation)* that you "know" and have control off—can then verify explicitly the performance of the sensor.
- Design the sensing task so that **state and measurement dynamics** are linear-Gaussian (LG), so that the **optimal (real-time) estimator** of the input signal can be explicitly constructed:

Kalman Filter

Kalman Filter – optimal estimator for Gaussian stochastic dynamics



Finger *actual* position:

$$\mathbf{x}_t = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

state

Sensor output:

$$\mathbf{z}_t = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

observation

Finger estimated position:

$$\tilde{\mathbf{x}}_t = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix}$$

estimator

Particular characteristics of **dynamics**

Model – how the sensor works.

Estimation scheme, potentially with memory: $\{\mathbf{z}_\tau\}_{\tau<t}$

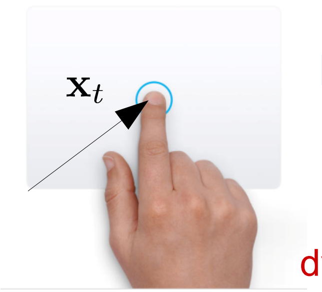
Optimal estimation scheme minimises the (time-) ***average Mean Squared Error*** :

$$\mathrm{MSE}(t) := \mathrm{Tr}\{\mathbf{\Sigma}_t\} = \mathrm{E}\big[(\mathbf{x}_t - \tilde{\mathbf{x}}_t)^T(\mathbf{x}_t - \tilde{\mathbf{x}}_t)\big] = \mathrm{E}\Big[|\mathbf{x}_t - \tilde{\mathbf{x}}_t|^2\Big]$$

(error) covariance matrix: $\mathbf{\Sigma}_t := \mathrm{E}\big[(\mathbf{x}_t - \tilde{\mathbf{x}}_t)(\mathbf{x}_t - \tilde{\mathbf{x}}_t)^T\big]$

[arXiv:1707.08131]

Kalman Filter – optimal estimator for Gaussian stochastic dynamics



$$\mathbf{x}_t = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \mathbf{z}_t = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \qquad \tilde{\mathbf{x}}_t = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix}$$

state **observation** **estimator**

dynamics → state → model → observation → estimation → estimator

Optimal estimator minimising (av.) MSE: **mean of the posterior distribution**

$$\tilde{\mathbf{x}}_{t+\delta t} = \int_t^{t+\delta t} D\mathbf{x}\, \mathbf{x}_{t+\delta t}\, p(\mathbf{x}_{t+\delta t} | \{\mathbf{z}_\tau\}_{\tau<t})$$

Special case of **linear Gaussian state and observation dynamics**:

$$d\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_t dt + \mathbf{G}_t d\mathbf{w}_t \qquad d\mathbf{z}_t = \mathbf{H}_t \mathbf{x}_t dt + d\mathbf{v}_t,$$

(Ito calculus)

Wiener (white-noise) terms: $\mathrm{E}[d\mathbf{w}_t d\mathbf{w}_s^T] = \mathbf{Q}_t\, \delta(t-s)\, dt \qquad \mathrm{E}[d\mathbf{v}_t d\mathbf{v}_s^T] = \mathbf{R}_t\, \delta(t-s)\, dt$

But, all the parameters are *a priori* known !!
(we are fighting "only" fluctuations)

→ "***waveform estimation***" (in contrast to "***tracking***")

[arXiv:1707.08131]

Kalman Filter – optimal estimator for Gaussian stochastic dynamics

Linear Gaussian (LG) state and observation dynamics:

$$\mathrm{d}\mathbf{x}_t = \mathbf{F}_t\mathbf{x}_t\mathrm{d}t + \mathbf{G}_t\mathrm{d}\mathbf{w}_t \qquad \mathrm{E}\left[\mathrm{d}\mathbf{w}_t\mathrm{d}\mathbf{w}_s^T\right] = \mathbf{Q}_t\,\delta(t-s)\,\mathrm{d}t$$

$$\mathrm{d}\mathbf{z}_t = \mathbf{H}_t\mathbf{x}_t\mathrm{d}t + \mathrm{d}\mathbf{v}_t, \qquad \mathrm{E}\left[\mathrm{d}\mathbf{v}_t\mathrm{d}\mathbf{v}_s^T\right] = \mathbf{R}_t\,\delta(t-s)\,\mathrm{d}t$$

Optimal estimator is provided by the solution to the **Kalman-Bucy equation** and… :

$$\frac{\mathrm{d}\tilde{\mathbf{x}}_t}{\mathrm{d}t} = \mathbf{F}_t\tilde{\mathbf{x}}_t + \mathbf{K}_t\left(\mathbf{z}_t - \mathbf{H}_t\tilde{\mathbf{x}}_t\right)$$

Kalman gain: $\mathbf{K}_t := \mathbf{\Sigma}_t\mathbf{H}_t^T\mathbf{R}_t^{-1}$

...**variance equation** for the **(error) covariance matrix**:

$$\frac{\mathrm{d}\mathbf{\Sigma}_t}{\mathrm{d}t} = \mathbf{F}_t\mathbf{\Sigma}_t + \mathbf{\Sigma}_t\mathbf{F}_t^T + \mathbf{G}_t\mathbf{Q}_t\mathbf{G}_t^T - \mathbf{\Sigma}_t\mathbf{H}_t^T\mathbf{R}_t^{-1}\mathbf{H}_t\mathbf{\Sigma}_t$$

Important facts:

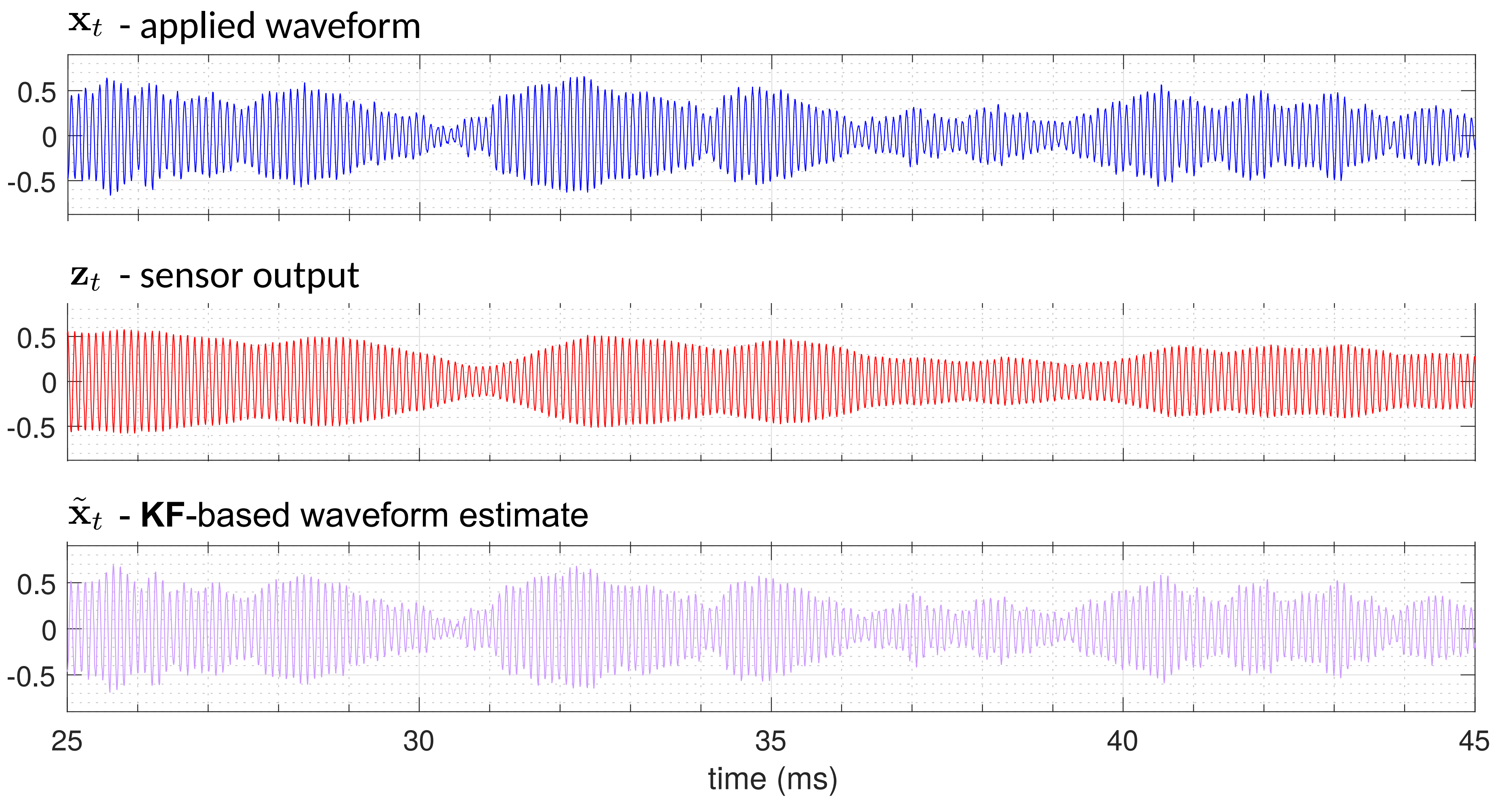
- *Kalman(-Bucy) Filter* (**KF**) is fast and causal – no need for *memory* of observations, $\{\mathbf{z}_\tau\}_{\tau<t}$, just the *last* one! The coviariance matrices, $\mathbf{\Sigma}_t$, can be precomputed!
- The KF provides the *error* for free, if the LG model assumed is correct…
- …, however, the stabilisation of the filter (convergence to the *steady-state solution*) serves as a verification tool.

Rudolf E. Kalman
"*The father of control theory*"

[R. E. Kalman and R. S. Bucy, J. Basic Eng. **83**, 95 (1961).]

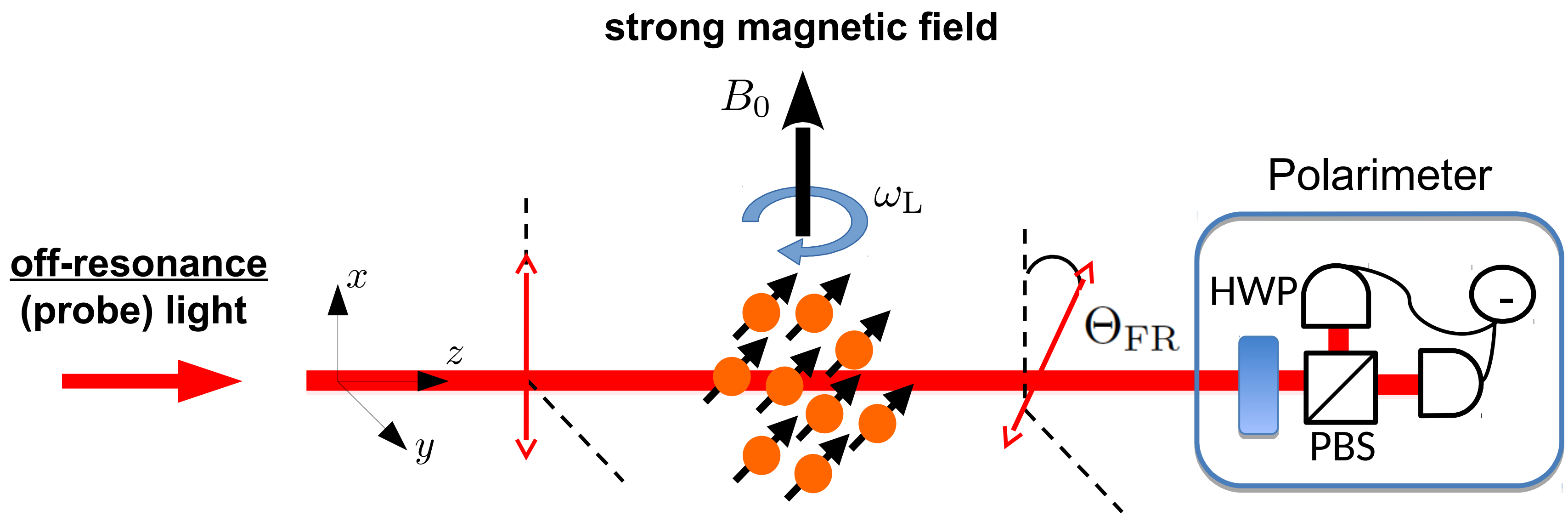
[arXiv:1707.08131]

Performance of our atomic sensor (SPOILER)



[arXiv:1707.08131]

Atomic sensor:
Monitoring spin precession using optical Faraday rotation



birefringence → phase shift → $\Theta_{FR} = g_D J_z$

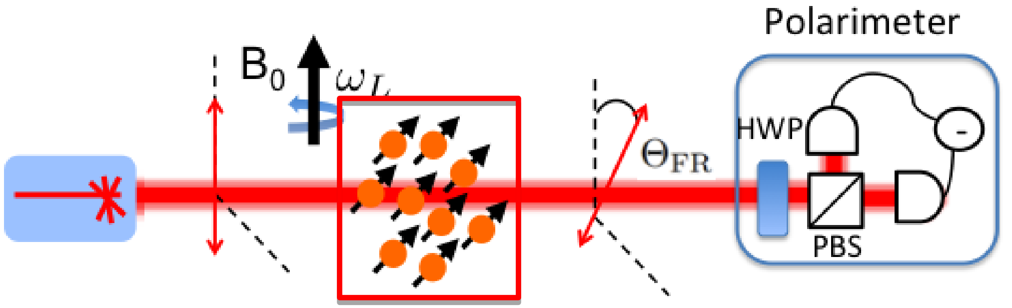
(far-detuned light)

NO MEASUREMENT BACK-ACTION!

$$\Theta_{FR} \approx \frac{c\, r_e\, f_{osc}}{A_{eff}} \frac{1}{(\nu - \nu'_\alpha)} J_z$$

[arXiv:1707.08131]

Experimental setup:

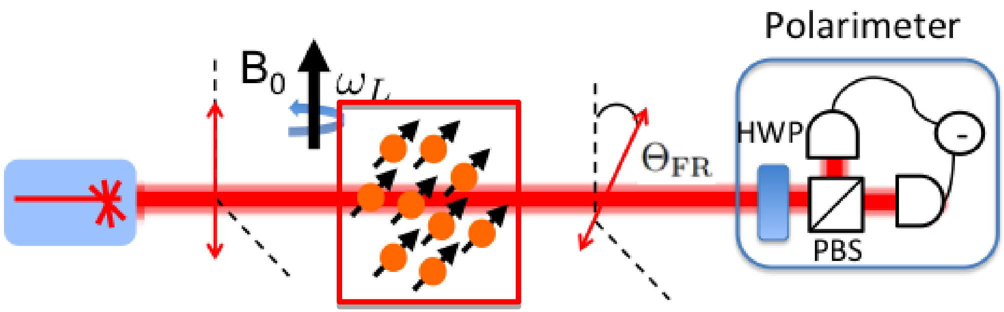


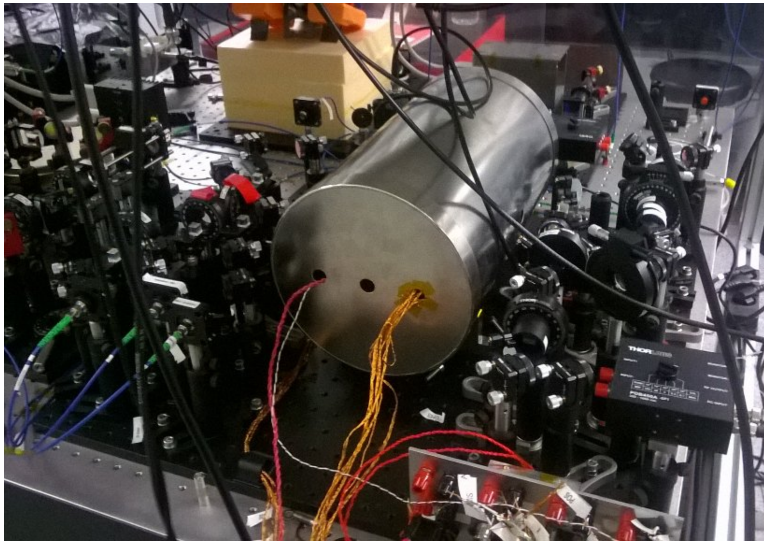
rubidium vapour cell

- ✓ Cell with Rb vapour and 100 Torr of N_2 buffer gas
- ✓ Cell is heated to reach densities: 10^{12} – 10^{13} atoms/cm^3

[arXiv:1707.08131]

Experimental setup:



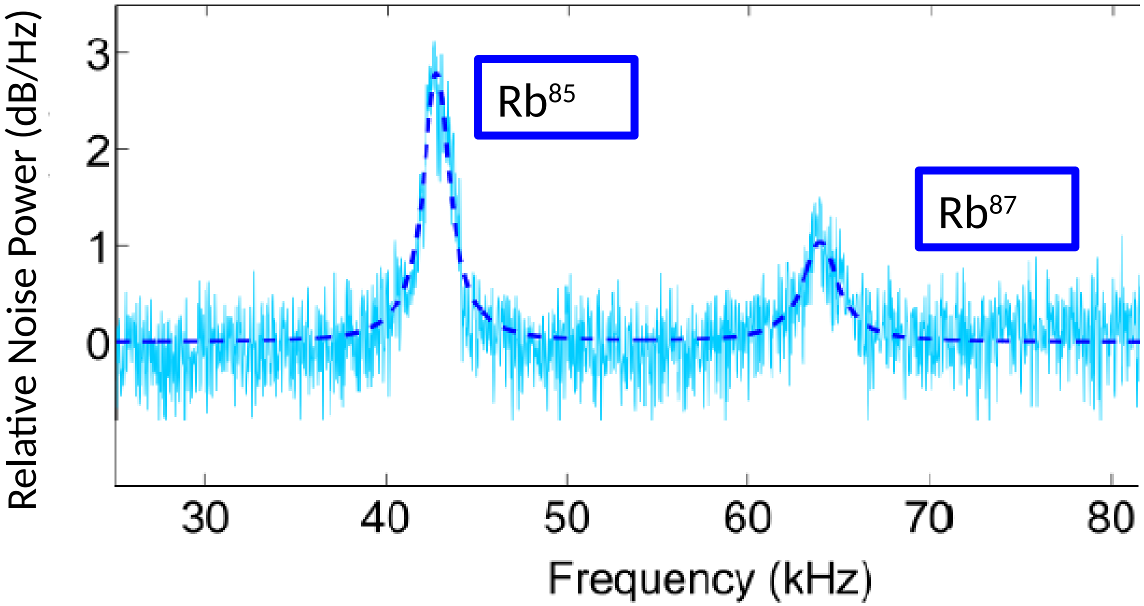


- ✓ Cell with Rb vapour and 100 Torr of N_2 buffer gas
- ✓ Cell is heated to reach densities: 10^{12} – 10^{13} atoms/cm^3
- ✓ Placed inside 1 layer of mu-metal shielding
- ✓ 3-axis DC-Fields & gradient coils in the beam propagation direction

[arXiv:1707.08131]

Noise spectroscopy of the sensor

Power noise spectrum of Rb in natural abundance



Fit spectrum to model:

$$S_{zz}(\omega) = S_{ph} + \frac{S_{at}}{(1/T_2)^2 + (\omega - \omega_0)^2}$$

[V. G. Lucivero et al. Phys. Rev. A, **95**, 041803 (2017)]

Power spectral density

Photon shot noise

Spin noise

$$S_{zz}(\omega) = S_{ph} + \frac{S_{at}}{(1/T_2)^2 + (\omega - \omega_0)^2}$$

Spin relaxation

Spin dynamics (Ornstein-Uhlenbeck process)

$$\mathrm{d}J_y(t) = \left(\omega_\mathrm{L} J_z(t) - \frac{1}{T_2} J_y(t)\right) \mathrm{d}t + \mathrm{d}W_y(t)$$

$$\mathrm{d}J_z(t) = -\left(\omega_\mathrm{L} J_y(t) - \frac{1}{T_2} J_z(t)\right) \mathrm{d}t + \mathrm{d}W_z(t)$$

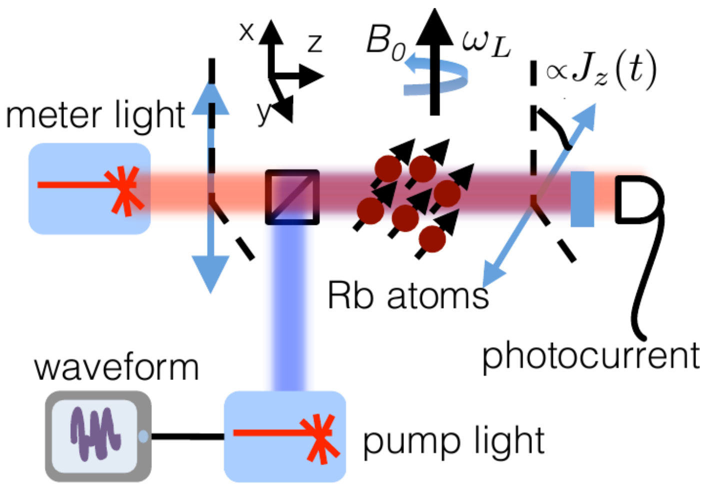
Spin measurement (via Faraday rotation)

$$I(t)\mathrm{d}t = g_\mathrm{D}\, J_z(t)\mathrm{d}t + \mathrm{d}W_\mathrm{D}(t)$$

[arXiv:1707.08131]

Our atomic sensor – a two-stage sensor

Waveform estimation – the waveform is encoded in the quadratures of the pump light



Signal (pump) – an O-U process (that we **know and control!**):

$$\mathrm{d}q(t) = -\kappa q(t)\mathrm{d}t + \mathrm{d}W_q(t),$$
$$\mathrm{d}p(t) = -\kappa p(t)\mathrm{d}t + \mathrm{d}W_p(t),$$

$$\mathbf{q}_t = [q(t), p(t)]^T$$

$$\mathcal{E}_\mathrm{P}(t) = [\cos(\omega_\mathrm{P} t), \sin(\omega_\mathrm{P} t)] \cdot \mathbf{q}_t$$

Atoms – O-U process from noise spectroscopy *plus* the pump term:

$$\mathrm{d}J_y(t) = \left(\omega_\mathrm{L} J_z(t) - \frac{1}{T_2} J_y(t)\right)\mathrm{d}t + \mathrm{d}W_y(t)$$

$$\mathbf{j}_t = [J_y(t), J_z(t)]^T$$

$$\mathrm{d}J_z(t) = -\left(\omega_\mathrm{L} J_y(t) - \frac{1}{T_2} J_z(t)\right)\mathrm{d}t + \mathrm{d}W_z(t) + \mathcal{E}_\mathrm{P}(t)\mathrm{d}t$$

Measurement (probe) – Faraday rotation with shot noise:

$$I(t_k) = g_\mathrm{D}\, J_z(t_k) + \xi_\mathrm{D}(t_k) \qquad \xi_\mathrm{D}(t_k) \equiv \frac{1}{\Delta}\int_{t_k-\Delta}^{t_k} \mathrm{d}w_\mathrm{sn}(t')$$

(finite sampling period)

Linear Gaussian state and observation dynamics:

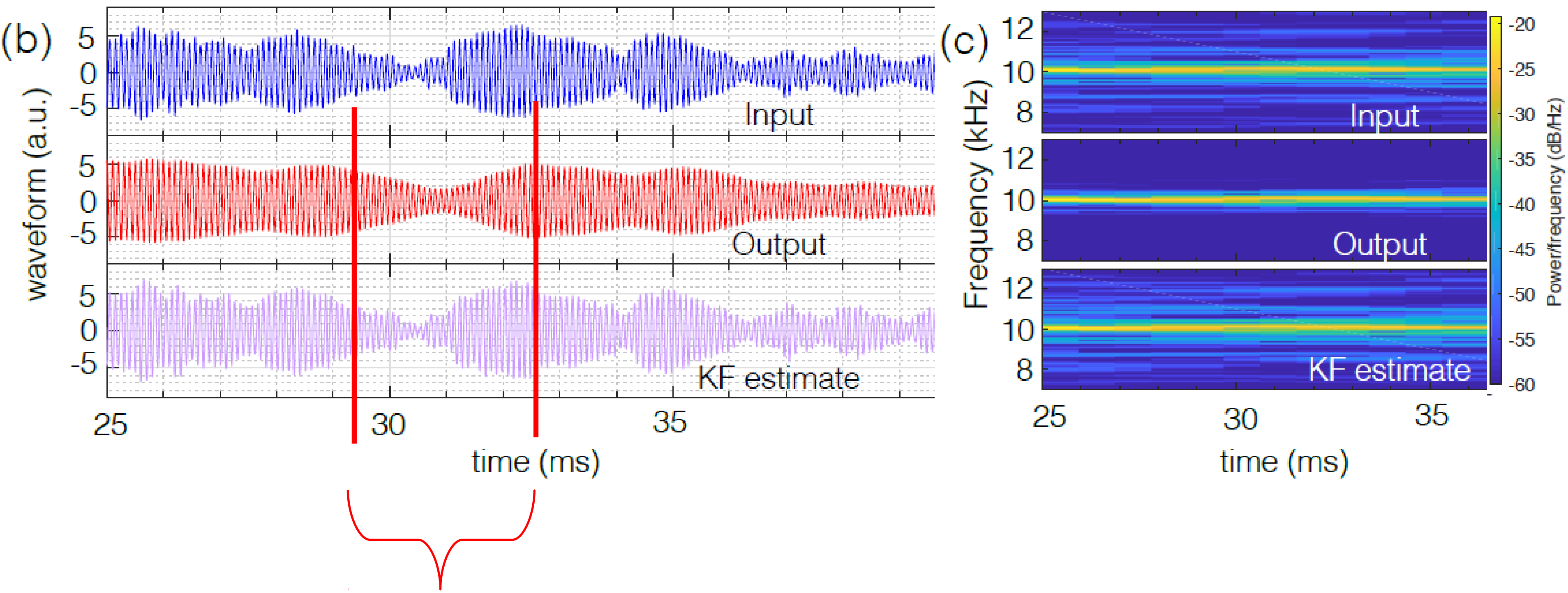
$$\mathrm{d}\mathbf{x}_t = \mathbf{F}_t \mathbf{x}_t \mathrm{d}t + \mathrm{d}\mathbf{w}_t \quad \rightarrow \quad \textbf{state: } \mathbf{x}_t = \mathbf{j}_t \oplus \mathbf{q}_t \quad \textbf{state noise: } \mathrm{d}\mathbf{w}_t = \mathrm{d}\mathbf{w}_t^{(\mathrm{J})} \oplus \mathrm{d}\mathbf{w}_t^{(\mathrm{q})}$$

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \quad \rightarrow \quad \textbf{observation: } \mathbf{z}_k \equiv z_k = I(t_k) \quad \textbf{observation noise: } \mathbf{v}_k \equiv v_k = \xi_\mathrm{D}(t_k)$$

Subtlety: we need to use **Hybrid** (Continuous-Discrete) **Kalman Filter**...

[arXiv:1707.08131]

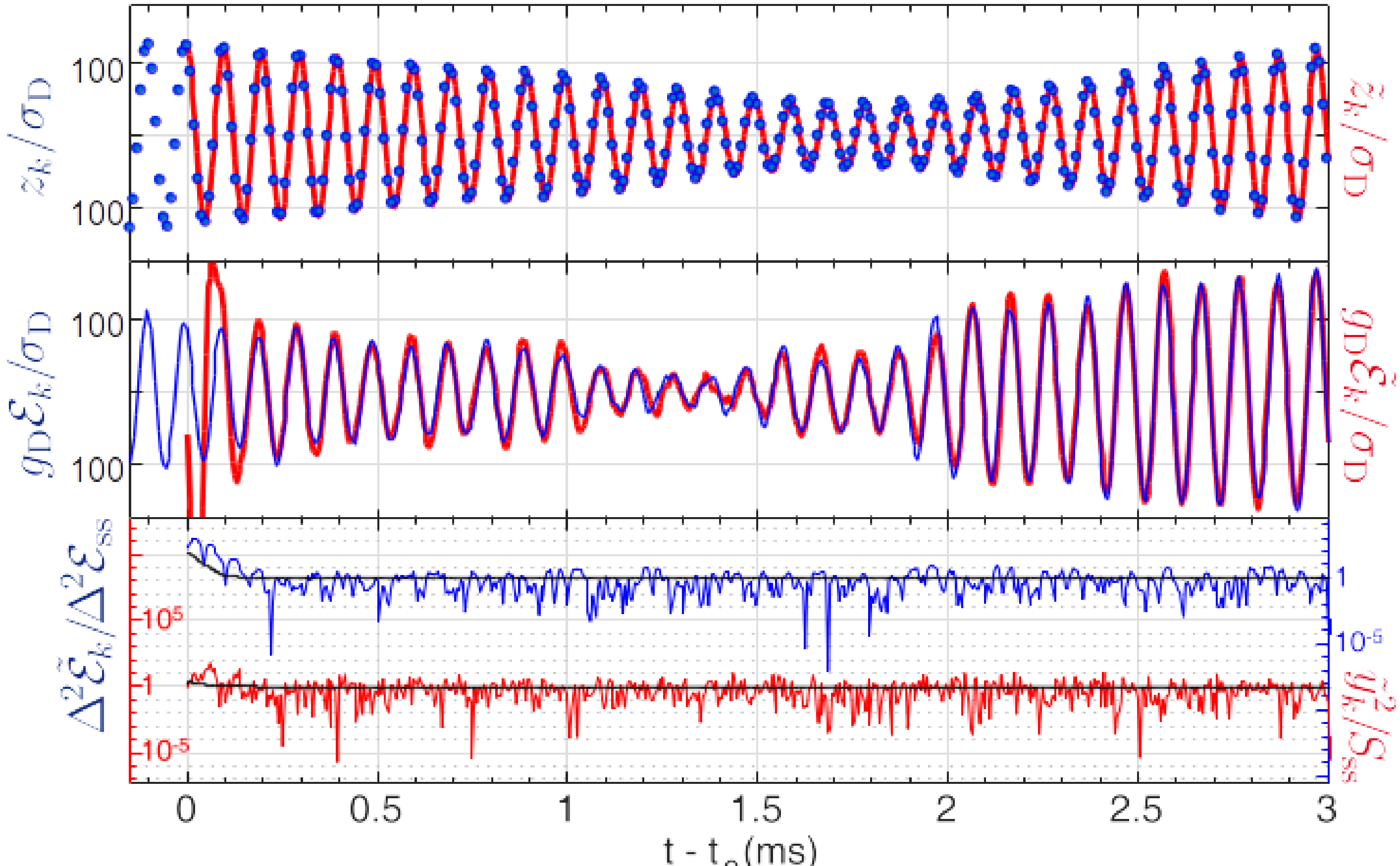
Validation of the Linear Gaussian model and the Kalman Filter performance



[arXiv:1707.08131]

Validation of the Linear Gaussian model and the Kalman Filter performance

True (blue) observation/waveform, *estimates* (red) of observation/waveform



Observation (**blue**) estimation error, *waveform* (**red**) estimation error

Kalman-filter construction provides also precision on estimates! (**black lines**)

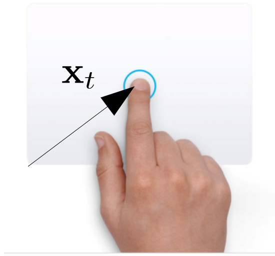
Verification: We focus on pump quadratures (not atoms) because we have access to true values!

[arXiv:1707.08131]

Tracking partially unknown signals with the atomic sensor and using the Kalman Filter

We track unknown **triangular waveforms** and employ (2^{nd} order) **kinematic model**

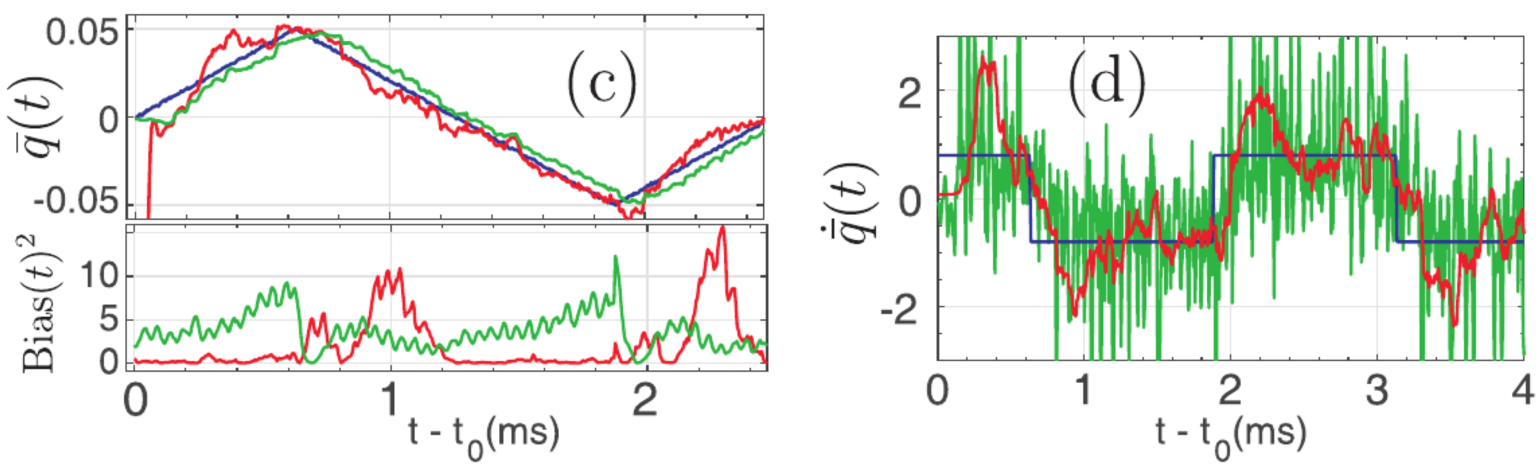
Approximate unknown dynamics by assuming that **acceleration** is constant on average but fluctuates.



$$\mathbf{q}_t = \begin{pmatrix} \mathbf{x}_t \\ \dot{\mathbf{x}}_t \\ \ddot{\mathbf{x}}_t \end{pmatrix} \qquad \mathrm{d}\mathbf{q}_t = \begin{pmatrix} \mathrm{d}\mathbf{x}_t \\ \mathrm{d}\dot{\mathbf{x}}_t \\ \mathrm{d}\ddot{\mathbf{x}}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{q}_t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{d}\mathbf{w}_t$$

(enlarge the state space)

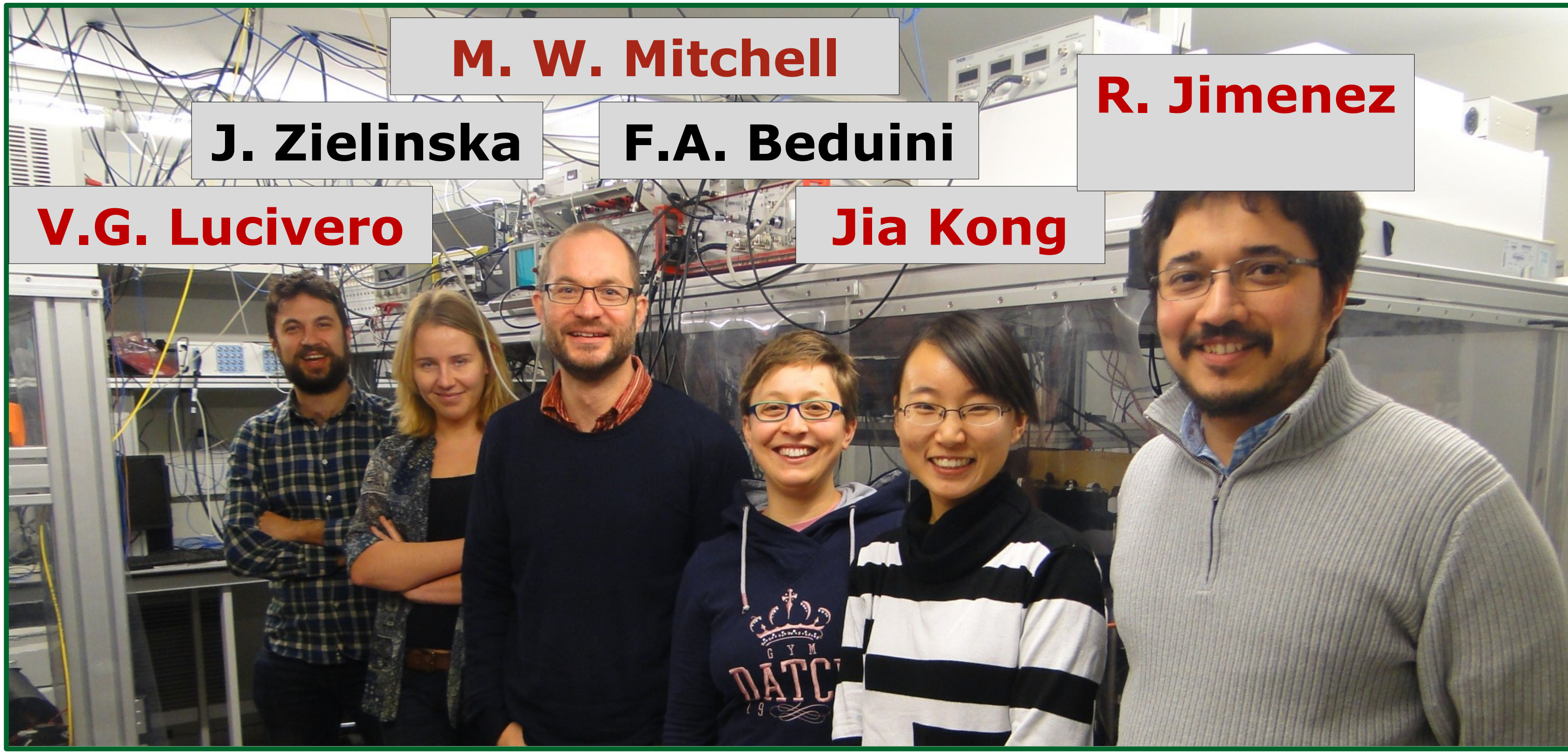
Tracking triangular waveforms – ***naive KF*** (green) and ***2^{nd} order kinematic model KF*** (red):

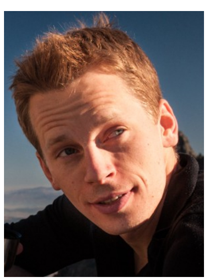


Model	Bias^2	Var	MSE
$\{\tilde{q}_k\}_{(\mathrm{WP})}$	3.36×10^{-5}	3.0×10^{-6}	3.66×10^{-5}
$\{\tilde{q}_k\}_{(\mathrm{PM})}$	1.02×10^{-5}	7.6×10^{-6}	1.78×10^{-5}

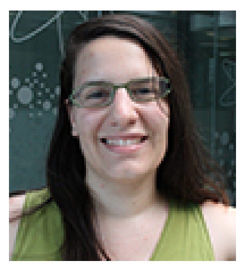
By enlarging the state space we increase noise (Var) in the estimate, but significantly decrease the Bias (*drag*) and, hence the MSE!

Thank You!





J. Kolodynski



C. Trollinou

A. Dimic